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Local symmetries of the non-Abelian two-form

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Abstract

An inhomogeneous gauge transformation law for non-Abelian two-forms in $B \wedge F$ type theories is proposed and corresponding invariant actions are discussed. The auxiliary one-form, required for maintaining vector gauge symmetry in some of these theories, transforms like a gauge field, and hence cannot be set to zero by a gauge choice. It can be set equal to the usual gauge field by a gauge choice, leading to gauge equivalences between different types of theories, those with the auxiliary field and those without. A new type of symmetry also appears in some of these theories, one which depends on local functions but cannot be generated by local constraints. The corresponding conserved currents and BRST charges are parametrized by the space of flat connections.

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1. Introduction

The non-Abelian two-form, or antisymmetric tensor potential, was introduced in the context of nonlinear σ-models [1–3] via an interaction term Tr $B \wedge F$. Here B is a two-form potential in the adjoint representation, and F is the field strength of the gauge field A. An action made up of this term alone is a Schwarz-type topological field theory [4–7]. It generalizes to four dimensions the familiar three-dimensional Chern–Simons action [8–10]. This action is diffeomorphism invariant, and serves as a toy model for some features of quantum gravity [11, 12]. Modifications involve adding a suitably chosen quadratic term, sometimes sacrificing diffeomorphism invariance, but leading to different physical theories. For example, Einstein gravity with a cosmological term can be recovered by adding a $B \wedge B$ term [13–15]. When the gauge group is taken to be $SU(N)$, modifications of the $B \wedge F$ action lead to Yang–Mills theory in a first-order formulation [16–19] or in a loop space formulation [20, 21]. In these models, the two-form appears as a field without its own dynamics. A theory where the two-form is dynamical can be constructed by introducing a kinetic term [22, 23]. This also happens to be a non-Abelian generalization of a mass generation mechanism for vector fields in four dimensions, which does not have a residual Higgs particle in the spectrum [24–29]. The corresponding non-Abelian theory appears to be stable under quantization upon using standard algebraic techniques of path integral quantization [30–32].

Despite its wide applicability, the nature of the non-Abelian two-form remains obscure. A two-form couples naturally to a world surface, so one possible description of this is as a gauge field for strings. This is a consistent description for the Abelian two-form. However, Teitelboim has argued [33] that 'surface-ordered' exponentials $\mathcal{P}\exp(-q\int B\,\mathrm{d}\Sigma)$ for the non-Abelian two-form cannot be defined in a reparametrization invariant fashion. The proof of this (and of all other results about the two-form) assumes that the two-form transforms homogeneously in the adjoint under local gauge transformations.

In this paper, I suggest that there is enough ambiguity in the local transformations of the two-form to allow an inhomogeneous gauge transformation law in place of the usual homogeneous one. It will still be possible to construct gauge invariant actions, some identical to known ones and some new ones. The dynamics and physical implications of these actions will be mentioned briefly, but will not be discussed in depth. Some of these actions are symmetric under local classical symmetries which are not generated by local constraints, a novelty for classical field theories. Some results of this paper have been reported briefly in [34], several details and new results are presented here.

In section 2 an inhomogeneous gauge transformation law is proposed for the non-Abelian two-form. Several actions invariant under this transformation are constructed in section 3. Vector gauge transformations are introduced in section 4 along with the auxiliary vector field, which transforms as a gauge field as well. A new class of transformations is also discussed there—local transformations which are not generated by local constraints. The connection-like nature of the auxiliary field allows an alternative set of transformation rules and actions for the non-Abelian two-form, discussed in section 5. These alternative rules and corresponding actions are shown to be equivalent to the earlier ones. The main results are summarized in section 6.

Notation: I shall use the notation of differential forms. The gauge connection one-form (gauge field) is defined in terms of its components as $A = -\mathrm{i}gA^a_\mu t^a\,\mathrm{d}x^\mu$, where t^a are the (Hermitian) generators of the gauge group satisfying $[t^a, t^b] = \mathrm{i}f^{abc}t^c$ and g is the gauge coupling constant. Any other coupling constant, which may be required in a given model, will be assumed to have been absorbed in the corresponding field. An example is a coupling constant m of mass dimension one, which appears in the action of some models as $mB \wedge F$, and may be absorbed into B. This will cause no problem since I am concerned solely with classical systems and classical symmetries. The gauge group will be taken to be $SU(N)$ for specificity. The gauge covariant exterior derivative of an adjoint p-form ξ_p will be written as

$$\mathrm{d}_A\,\xi_p \equiv \mathrm{d}\,\xi_p + A \wedge \xi_p + (-1)^{p+1}\xi_p \wedge A \tag{1}$$

where d stands for the usual exterior derivative. The field strength is $F = \mathrm{d}A + A \wedge A$, and satisfies the Bianchi identity, $\mathrm{d}F + A \wedge F - F \wedge A = 0$. Under a gauge transformation, the gauge field transforms as $A \to A' = UAU^\dagger - \mathrm{d}UU^\dagger$. For brevity, I will write $\phi \equiv -\mathrm{d}UU^\dagger$. Note that ϕ is a flat connection, $\mathrm{d}\phi + \phi \wedge \phi = 0$.

2. Gauge symmetries

In this section, I will construct a modification of $SU(N)$ gauge transformation rules for the non-Abelian two-form, starting from the action $\int \mathrm{Tr}\,B \wedge F$. In terms of its components,

$B = -\frac{1}{2}\mathrm{i}g B^a_{\mu\nu} t^a \,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. Under a local $SU(N)$ transformation represented by U, the gauge field A and the field strength F transform as

$$A \to A' = UAU^\dagger + \phi \qquad F \to F' = UFU^\dagger. \tag{2}$$

Invariance of the action under local $SU(N)$ transformation is usually enforced by assuming that B transforms homogeneously in the adjoint,

$$B \to B' = UBU^\dagger. \tag{3}$$

In addition to $SU(N)$ gauge transformations, the action is invariant under a non-Abelian generalization of Kalb–Ramond gauge transformation [35],

$$B' = B + \mathrm{d}_A\xi \qquad A' = A \tag{4}$$

where ξ is an arbitrary one-form. Because of Bianchi identity, the Lagrangian changes by a total divergence under this transformation. This will be referred to as vector gauge transformation; the name Kalb–Ramond transformation will be reserved for the case where d appears instead of d_A in (4). The group of vector gauge transformations is Abelian; two such transformations with parameters ξ_1 and ξ_2 combine to yield a transformation with parameter $\xi_1 + \xi_2$. The two types of transformations are independent of each other, and therefore combine as

$$A' = UAU^\dagger + \phi \qquad B' = UBU^\dagger + \mathrm{d}_{A'}\xi' \tag{5}$$

provided that the one-form ξ transforms homogeneously in the adjoint as $\xi' = U\xi U^\dagger$.

The gauge transformation law for B, as given in (3), is not unique because of vector gauge transformations. It is clear that if the two types of transformations are independent, ξ may not transform like a connection, i.e. as $\xi' = U\xi U^\dagger + \phi$. This is because connections do not form a group under addition. But it is possible to choose a connection in place of ξ, and construct novel symmetries which mix the two types of transformations. There is such a connection which depends on U, the flat connection $\phi = -\mathrm{d}UU^\dagger$. When ϕ is inserted in place of ξ', equation (5) is modified to

$$A' = UAU^\dagger - \mathrm{d}UU^\dagger \qquad B' = UBU^\dagger + UA \wedge \mathrm{d}U^\dagger - \mathrm{d}U \wedge AU^\dagger - \mathrm{d}U \wedge \mathrm{d}U^\dagger. \tag{6}$$

The $SU(N)$ transformation U is completely arbitrary, but there is no (other) arbitrary vector field involved in this equation. So this no longer has anything to do with the vector gauge transformations. In fact, it is easy to see that (6) is nothing but a modification of the gauge transformation law for B. To see this, it is sufficient to show that two successive gauge transformations of B combine according to the group multiplication law of $SU(N)$.

Consider two gauge transformations U_1 and U_2, applied successively to the fields. According to (6), the fields transform under U_1 as

$$A_1 = U_1 A U_1^\dagger + \phi_1 \qquad B_1 = U_1 B U_1^\dagger + \mathrm{d}_{A_1}\phi_1 \tag{7}$$

with $\phi_1 = -\mathrm{d}U_1 U_1^\dagger$, and then under U_2 as

$$A' = U_2 A_1 U_2^\dagger + \phi_2 \qquad B' = U_2 B_1 U_2^\dagger + \mathrm{d}_{A_2}\phi_2 \tag{8}$$

with $\phi_2 = -\mathrm{d}U_2 U_2^\dagger$. Substituting for A_1 and B_1 in (8) their expressions from (7), I get back (6), but with $U = U_2 U_1$. The transformation is clearly invertible and continuously connected to the identity. So it is perfectly acceptable to treat equation (6) as the $SU(N)$ gauge transformation law for the fields, i.e. to replace (3) by (6). The action changes only by a total divergence,

$$\mathrm{Tr}\, B' \wedge F' = \mathrm{Tr}\, B \wedge F + \mathrm{d}\ \mathrm{Tr}(\phi \wedge UFU^\dagger). \tag{9}$$

The gauge transformation law of A is of course the standard one, but that of B is quite unusual. The fact that it is inhomogeneous makes B appear more like a connection than is usually thought. For an Abelian gauge group, all commutators vanish, so $B' = B$ just as it should be. Further, equation (6) makes sense only if B is a two-form and A is a one-form, and therefore is a symmetry of the action only in four dimensions. However, it is possible to construct similar transformations for higher p-forms in $p + 2$ dimensions. For example, a three-form B_3 coupled to the gauge field via a $B_3 \wedge F$ interaction in five dimensions can be taken to transform under the gauge group as

$$B_3' = UB_3U^\dagger + UAU^\dagger \wedge \phi \wedge \phi - \phi \wedge \phi \wedge UAU^\dagger. \tag{10}$$

The corresponding variation in the action is then

$$\delta \int \mathrm{Tr}\, B_3 \wedge F = \int \mathrm{d}\,\mathrm{Tr}(\mathrm{d}\phi \wedge UFU^\dagger). \tag{11}$$

It is possible to consider other generalizations of the transformation law in place of equation (10), but three-forms or higher p-forms and corresponding higher dimensional actions will not be explored in this paper. One can also consider a non-trivial topology for spacetime, in which case the objects A, ϕ and B are not globally defined in general. In such cases, the modified gauge transformations relate topological charges of A and B, and may lead to novel descriptions of non-perturbative objects.

For perturbation theory, this modification does not pose any major problem. This is because the inhomogeneous part is in some sense strictly finite, becoming irrelevant for gauge transformations infinitesimally close to the identity. On the other hand, infinitesimal transformations are all that are needed for an analysis using the Becchi–Rouet–Stora–Tyutin (BRST) differential. Let me write the transformation law for B with Lorentz indices and coupling constants restored,

$$B'_{\mu\nu} = UB_{\mu\nu}U^\dagger - [UA_{[\mu}U^\dagger, \partial_{\nu]}UU^\dagger] + \frac{\mathrm{i}}{g}[\partial_\mu UU^\dagger, \partial_\nu UU^\dagger] \tag{12}$$

where I have written $A_\mu = A^a_\mu t^a, B_{\mu\nu} = B^a_{\mu\nu}t^a$, etc. The BRST transformations corresponding to the gauge transformations (6) are

$$\begin{aligned} &sA^a_\mu = \partial_\mu\omega^a + gf^{abc}A^b_\mu\omega^c \qquad sB^a_{\mu\nu} = gf^{abc}\left(B^b_{\mu\nu}\omega^c + A^b_{[\mu}\partial_{\nu]}\omega^c\right) \\ &s\omega^a = -\tfrac{1}{2}gf^{abc}\omega^b\omega^c. \end{aligned} \tag{13}$$

This BRST operator is nilpotent, $s^2 = 0$, as a BRST operator should be. Comparison with the conventional BRST rules [17, 30–32, 36] shows that the inhomogeneous part of the gauge transformation law of B is like the vector gauge transformation with $\partial_\mu\omega^a$ playing the role of the vector parameter. This is why known perturbative calculations, which include the vector gauge transformations in the analysis, will not need much modification. But quite clearly, it is not correct to think of the new gauge transformation as a special case of vector gauge transformation with parameter $\partial_\mu\omega^a$, since one cannot start from the latter and reach equation (6).

The total divergence which appears in the variation of the action will contribute to the conserved current of gauge symmetry. Consider an $SU(N)$ gauge transformation $U = 1 + \mathrm{i}g\xi^a t^a$ infinitesimally close to the identity. The corresponding change in the action is

$$\delta \int \frac{1}{4}\epsilon^{\mu\nu\rho\lambda}B^a_{\mu\nu}F^a_{\rho\lambda} = -\frac{1}{2}\int \epsilon^{\mu\nu\rho\lambda}\partial_\mu\left(\xi^a\partial_\nu F^a_{\rho\lambda}\right). \tag{14}$$

It follows that the conserved current of gauge symmetry has a *topologically conserved* component $j^{a\mu}_T = -\frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu F^a_{\rho\lambda}$. This looks like a current of non-Abelian Dirac monopoles. This current is not gauge covariant, but in a configuration where F vanishes on the boundary (e.g. Euclidean finite action) this makes a vanishing contribution to the conserved charge.

The appearance of the connection in the symmetry transformation law for another field might be expected when the connection describes transport on a curved spacetime. For gauge theories, the explanation is similar. If there is any dynamics at all, the theory must contain gauge covariant derivatives, and therefore a gauge field. Even when this gauge field is flat, i.e. it could be made to vanish in some gauge, there will be an inhomogeneous transformation of B. This point is relevant for the naive generalization of duality between a two-form and a scalar [37, 38], or outside the horizon of black holes with a non-Abelian topological charge [22]. The modification of the gauge transformation law for B does not affect the duality relation or the topological charge in these cases.

3. Symmetric actions

It is now easy to construct several novel actions involving A and B, with interesting physical implications, by demanding invariance only under the gauge transformation (6), and ignoring vector gauge transformations for the moment. One reason for this exercise is to show that an inhomogeneous transformation law for B does not automatically rule out construction of invariant actions. Another reason will become apparent later when I argue that some of the actions constructed below are gauge-fixed versions of actions with explicit vector gauge symmetry. Note that

$$\mathrm{d}A' = -\phi \wedge UAU^{\dagger} + U\mathrm{d}AU^{\dagger} - UAU^{\dagger} \wedge \phi + \mathrm{d}\phi \tag{15}$$

so that both the combinations $(B + \mathrm{d}A)$ and $(B - A \wedge A)$ transform covariantly under the gauge group. An interaction Lagrangian of the form $\mathrm{Tr}(B + \mathrm{d}A) \wedge F$ or $\mathrm{Tr}(B - A \wedge A) \wedge F$ will be exactly invariant under gauge transformations, and the conserved current for $SU(N)$ gauge transformations will be gauge covariant. These actions differ from $\mathrm{Tr}\, B \wedge F$ by a total divergence, and violate CP.

Invariant actions can be built by starting from one of these forms of the interaction term. Then a quadratic term, constructed out of $(B + \mathrm{d}A)$, or the covariant field strength $(\mathrm{d}_A B - \mathrm{d}F)$, can be added for an action invariant under the modified gauge transformations. For example, a class of CP-violating actions come from using quadratic terms of the type $B \wedge B$, as in

$$S_1 = \int \mathrm{Tr}\left(B \wedge F - \frac{1}{2}(B + \mathrm{d}A) \wedge (B + \mathrm{d}A)\right) \tag{16}$$

which is equivalent to a total divergence after B is eliminated [39]. Obviously, there are many variations on this theme which arise from replacing B in $B \wedge F$ by either of the two gauge covariant combinations and from using $(B - A \wedge A)$ in one or both factors of the second term. All these actions are classically equivalent, i.e. they differ by total divergences. They are also equivalent in the path integral, after Gaussian integration over B, to a total divergence. All the information about these theories resides on the boundary. An interesting point is that some actions of this CP-violating class vanish altogether upon using the equation of motion for B. An example is

$$S_1' = \int \mathrm{Tr}((B - A \wedge A) \wedge F - (B + \mathrm{d}A) \wedge (B - A \wedge A)). \tag{17}$$

An action belonging to another class is a first-order formulation of Yang–Mills theory, along the lines of [17–19], but without vector gauge symmetry,

$$S_2 = \int \mathrm{Tr}\left(B \wedge F + \frac{1}{2}(B + \mathrm{d}A) \wedge *(B + \mathrm{d}A)\right). \tag{18}$$

The equation of motion for B is $B + \mathrm{d}A = *F$, which can be put back into the action. This action is then classically equivalent, i.e. equal up to a total divergence, to Yang–Mills theory.

Again, there are several variations on this theme, but not all of them have local dynamics. One example of this type is

$$S_2' = \int \mathrm{Tr}(B \wedge F + (B + \mathrm{d}A) \wedge *(B - A \wedge A)) \tag{19}$$

which is equivalent to a total divergence. So even though the action (19) written in terms of B seems to require a metric, actually the dynamics it describes is independent of the metric and lives fully on the boundary.

Another type of action involves a gauge covariant field strength for B. This can be defined as $\tilde{H} = \mathrm{d}_A(B + \mathrm{d}A) \equiv \mathrm{d}_A(B - A \wedge A) = \mathrm{d}_A B - \mathrm{d}F$. Like other gauge covariant combinations mentioned in this section, this field strength is not invariant under vector gauge transformations. With the help of this field strength, I can write down an action in which the two-form B is dynamical,

$$S_3 = \int \mathrm{Tr}\left(\frac{1}{2}\tilde{H} \wedge *\tilde{H} + \frac{1}{2}F \wedge *F + B \wedge F\right). \tag{20}$$

The B-independent part of this action is a nonlinear σ-model for the gauge field A. The term quadratic in derivatives of A reads, with gauge and Lorentz indices restored,

$$-\tfrac{1}{4}\left(\delta^{bd} + g^2 f^{abc} f^{ade} A^c_\lambda A^{\lambda e}\right) \partial_{[\mu} A^b_{\nu]} \partial^{[\mu} A^{\nu]d} - \tfrac{1}{2} g^2 f^{abc} f^{ade} \partial_{[\mu} A^b_{\nu]} A^c_\lambda \partial^{[\nu} A^{\lambda]d} A^{\mu e}. \tag{21}$$

This action is not power counting renormalizable, nor should it lead to a consistent quantum theory, like any nonlinear σ-model in four dimensions. This action also ignores vector gauge transformations, but I will argue in the next section that it is gauge equivalent to the theory with an auxiliary field and explicit vector gauge symmetry. In the Abelian limit where the structure constants vanish, $\tilde{H}$ becomes the usual field strength for a set of Abelian two-forms. This action then describes a set of topologically massive Abelian gauge fields.

Many other actions can be constructed using (6) as the gauge transformation law and these will correspond to different physical systems. One can even construct actions which do not have the $B \wedge F$ term as the cornerstone. I will not explore such actions here. Finally, for p-forms in $p + 2$ dimensions, a generalized $SU(N)$ gauge transformation law can be trivially constructed by demanding covariance of

$$B_p - A \wedge A \wedge \cdots \wedge A \qquad (p \text{ factors}). \tag{22}$$

Other generalizations for higher p-forms ($p > 2$) are clearly possible, including construction of a 'tower' of $1, \ldots, p$ forms.

4. Auxiliary one-form

The actions mentioned in the previous section, except for the pure $B \wedge F$ action and its alternatives, were not symmetric under vector gauge transformations. These transformations need to be re-introduced into the discussion. The interaction Lagrangian $\mathrm{Tr}\, B \wedge F$ is symmetric up to a total divergence under the vector gauge transformations (4). Any term quadratic in B, including a possible kinetic term, is not invariant. This is obvious for quadratic terms like $B \wedge B$ or $B \wedge *B$. The 'field strength' $\mathrm{d}_A B$ also changes under these transformations,

$$\mathrm{d}_A B \to \mathrm{d}_A B + F \wedge \xi - \xi \wedge F. \tag{23}$$

Indeed a no-go theorem [40] asserts that a kinetic term for the two-form, invariant under both types of gauge transformations, cannot be constructed unless additional fields are introduced. Note that since this theorem was proved by using the BRST structure of these theories, the modification of the gauge symmetry displayed in the previous section does not change its proof.

Some actions which use auxiliary one-form fields to compensate for the vector gauge transformations have been known for some time [17, 23]. In these actions, a one-form field C is introduced, and is assumed to shift under these transformations,

$$A' = A \qquad B' = B + \mathrm{d}_A\xi \qquad C' = C + \xi. \tag{24}$$

The combination $(B - \mathrm{d}_A C)$ clearly remains invariant under these transformations, as does the compensated field strength

$$H = \mathrm{d}_A B - F \wedge C + C \wedge F. \tag{25}$$

How does C behave under local $SU(N)$ transformations? Clearly, it has to transform in the adjoint representation. There is now no need for $\mathrm{d}A$ to cancel the inhomogeneous part of gauge transformations of B. Instead, the auxiliary field C can be taken to transform inhomogeneously, like a connection, under the gauge group,

$$C' = UCU^\dagger + \phi. \tag{26}$$

With this choice, the combination $(B - \mathrm{d}_A C)$ transforms covariantly under the gauge group, $(B - \mathrm{d}_A C) \to U(B - \mathrm{d}_A C)U^\dagger$. The field strength H also transforms covariantly, $H \to UHU^\dagger$. Both these combinations, $(B - \mathrm{d}_A C)$ and H, are also invariant under the vector gauge transformations in equation (24), with ξ transforming homogeneously in the adjoint, $\xi' = U\xi U^\dagger$. Therefore, terms quadratic in $(B - \mathrm{d}_A C)$ or H can be used for construction of symmetric actions, invariant under both usual and vector gauge transformations. These actions are the same as those discussed in [17–19, 23, 30–32]. So for the same actions, one can choose B and C either to transform homogeneously or not, but the choice of inhomogeneity leads to some interesting new results, as will be seen shortly.

Since the flat connection $\phi = -\mathrm{d}UU^\dagger$ does not transform homogeneously under the gauge group, but the vector parameter ξ has to do so, it is clear that equation (26) cannot be a special case of vector gauge transformations, despite the formal similarity. Nor is it possible to take the vector parameter ξ to transform like a connection, because connections do not add, so vector gauge transformations will not form a group. This also provides an additional reason why it is not possible to set $C = 0$ by a gauge choice. Since C transforms like a connection, even if it is made to vanish in one gauge, it will be non-zero upon an $SU(N)$ gauge transformation.

The BRST transformations for C include additional ghosts due to the vector gauge transformations [32], but the part that comes from $SU(N)$ transformations is now

$$sC^a_\mu = \partial_\mu\omega^a + gf^{abc}C^b_\mu\omega^c. \tag{27}$$

The full BRST operator is nilpotent, $s^2 = 0$, and all calculations which use BRST analysis will go through. However, for finite transformations, C is now a connection under the gauge group, and the vector gauge transformation (24) shifts it by the *difference* ξ of any two connections.

There is a further global symmetry of theories which contain B only in the combination $\mathrm{d}_A B$ and in the term $B \wedge F$. If B is shifted by a constant multiple of F, $\mathrm{d}_A B$ remains invariant by Bianchi identity, while $B \wedge F$ changes by a total divergence. This shift is independent of vector gauge transformations, since there is no ξ for which $F = \mathrm{d}_A\xi$. But when this is combined with a special type of vector gauge transformation, the result is a novel type of local symmetry. Consider a local $SU(N)$ matrix $\tilde{U}$, and construct the flat connection $\tilde{\phi} = -\mathrm{d}\tilde{U}\tilde{U}^\dagger$. Symmetry under (24) requires only the difference ξ between any two connections, or a constant multiple of it, as the transformation parameter. Choose $\xi = \text{constant} \times (A - \tilde{\phi})$.

In combination with the shift, this choice of ξ produces a completely new type of symmetry transformation,

$$A' = A \qquad B' = B + \alpha(A - \tilde{\phi}) \wedge (A - \tilde{\phi}) \qquad C' = C + \alpha(A - \tilde{\phi}) \quad (28)$$

with α an arbitrary constant and $\tilde{\phi}$ an arbitrary flat $SU(N)$ connection. The compensated field strength H of (25) remains invariant, while the $B \wedge F$ term changes by a total divergence,

$$\begin{aligned} &\delta \int \mathrm{Tr}\, B \wedge F = \alpha \int \mathrm{d}\,\mathrm{Tr}\left(\frac{1}{3} A \wedge A \wedge A - \tilde{\phi} \wedge F\right) \\ &\delta(B - \mathrm{d}_A C) = -\alpha F \qquad \delta H = 0. \end{aligned} \quad (29)$$

It follows that an action containing the field strength H and the interaction $B \wedge F$ will be invariant (up to a total divergence) under this set of transformations.

Why is this a new type of symmetry? This seems to be only a local $SU(N)$ transformation, similar to gauge transformations, combined with global shift of variables. However, in general local transformations are generated by local first-class constraints. Vector gauge transformations are local, and they are generated by local constraints [41, 42]. On the other hand, the shift $B \to B + \alpha F$ is a *global* transformation. The transformations (28) are a combination of the two, but cannot be generated by local constraints. The simplest way to see this is by noting that the local part of these transformations is not connected to the identity; that is, for infinitesimal $\tilde{\phi}$, or $\tilde{U}$ infinitesimally close to the identity, equation (28) cannot be written as transformations which are themselves infinitesimally close to the identity transformation. It follows that (28) cannot be generated by any constraint. It also means that this symmetry will not appear in the BRST charge.

This can be seen directly in the BRST approach, starting with the transformation for infinitesimal $\tilde{\phi}$. For $\tilde{U} = 1 + \mathrm{i}g\delta\lambda\tilde{\theta}^a t^a$, with $\delta\lambda$ an anticommuting constant and $\tilde{\theta}^a$ an anticommuting field (that would be ghost), I can write the infinitesimal changes in the fields as follows from (28),

$$\delta B = \alpha A \wedge A + \alpha\delta\lambda(\mathrm{d}\tilde{\theta} \wedge A + A \wedge \mathrm{d}\tilde{\theta}) \qquad \delta C = \alpha(A + \delta\lambda \mathrm{d}\tilde{\theta}) \quad (30)$$

where $\mathrm{d}\tilde{\theta} = \mathrm{i}gt^a\partial_\mu\tilde{\theta}^a \mathrm{d}x^\mu$. Obviously the derivative $\delta/\delta\lambda$, which would be part of the BRST operator, does not have any meaning. This is because α has been treated as a finite constant, so perhaps it should be replaced by $\delta\lambda\alpha$, where α is now an anticommuting constant. But this will clearly not produce the correct BRST operator either, because then the $\tilde{\theta}$-dependent terms disappear from equation (30). So there is no BRST construction which includes this symmetry, which is another way of showing that it cannot be generated by local constraints, since the BRST charge has to include all local constraints. The transformations of equation (28) should not therefore be confused with usual local symmetry transformations. These should properly be called *semiglobal* transformations, elements of a class of global transformations, parametrized by both the global parameter α and local $SU(N)$ matrices $\tilde{U}$.

Since the BRST charge is fundamental to the local Hamiltonian quantization of gauge theories (see e.g. [43, 44]), a possible interpretation of the failure to construct a BRST charge is that the quantum theory of this system must include non-local objects and operators with appropriate induced actions of the symmetry group, in order to maintain the classical symmetries. This is not very surprising since a two-form naturally couples to world sheets, so it can govern the dynamics of fields living on one-dimensional objects.

The fact that the auxiliary connection C can be shifted by the difference of two connections also provides a way of relating actions constructed in this section with the help of C and those of the previous section, constructed without C and without vector gauge symmetry. Consider the case where these two connections are C and A, i.e. consider a vector gauge transformation with $\xi = \alpha(A - C)$ where α is some constant. The transformed fields are

$$B' = B + \alpha F + \alpha A \wedge A - \alpha \mathrm{d}_A C \qquad C' = (1-\alpha)C + \alpha A. \tag{31}$$

The compensated field strength H remains invariant, as it should under a vector gauge transformation, but in addition it has a familiar form for $\alpha = 1$ in terms of the transformed fields. If I choose $\alpha = 1$ in (31), the two connections become related to each other by $C' = A$, and then the field strength is $H = \mathrm{d}_A B' - F \wedge A + A \wedge F$, which because of the Bianchi identity can be written as $H = \mathrm{d}_A B' - \mathrm{d}F$. This has the same form as the field strength defined in the previous section, before the introduction of the auxiliary one-form. So even though C cannot be set to vanish in (25) by a gauge choice (because a subsequent $SU(N)$ transformation will make it non-vanishing), it can be absorbed into B in the above sense, in which it is 'replaced' by A. Note also that for $\alpha = 1$, the invariant combination $(B - \mathrm{d}_A C)$ can be written as

$$B' - \mathrm{d}_{A'} C' = B' - \mathrm{d}_A A = B' - F - A \wedge A. \tag{32}$$

This links the actions mentioned in the previous section with those mentioned here. Finally, note that C can also be 'replaced' by an arbitrary flat connection via a vector gauge transformation.

5. Two connections for two-form

In the previous section, it was shown that the auxiliary vector field C transforms like a gauge field under usual gauge transformations, and shifts by the 'difference of two connections' under a vector gauge transformation. It was also shown that by an appropriate vector gauge transformation, the auxiliary field could be 'shifted away' to be replaced, in a manner, by the gauge field A. In this section, the picture of C as a second connection in the theory will be made more explicit, and more actions will be shown to be related by gauge symmetry to those already mentioned.

The starting point is the observation that the auxiliary connection C always appears in conjunction with the two-form B. In fact, it is needed for all actions with vector gauge symmetry, except for the pure $B \wedge F$ action. But even this form of the action need not be treated as sacrosanct, since C can be absorbed in B in a specific manner using vector gauge transformations. Then I can conclude that the non-Abelian two-form B and the auxiliary connection C cannot be separately included in any theory, but has to be considered as a pair.

Is it then possible to formulate the gauge transformation laws of the pair (B, C) purely in terms of themselves without referring to the usual gauge field A as was done earlier? This would remove the dependence of B on the gauge field A, which is somewhat artificial, since there is no converse dependence—a theory of the gauge field A can be defined without invoking B. If (6) can be described as being in a special gauge for the vector gauge transformations (24), it may be possible to remove the A dependence in the transformation of B.

Following this argument, let me first define a new set of $SU(N)$ gauge transformations for A, B and C as

$$\begin{aligned} &A' = UAU^\dagger + \phi \qquad B' = UBU^\dagger + UCU^\dagger \wedge \phi + \phi \wedge UCU^\dagger + \phi \wedge \phi \\ &C' = UCU^\dagger + \phi. \end{aligned} \tag{33}$$

These are the same as the rules of (6) and (26), but with the gauge field C taking the place of the gauge field A in the transformation rule for B. It will be shown later that actions invariant under (33) are in fact equivalent to actions mentioned in previous sections.

Just as the earlier transformations, these combine according to the group multiplication law of $SU(N)$. This can be seen by applying two successive $SU(N)$ gauge transformations U_1 and U_2. One immediate consequence of choosing (33) as the gauge transformation law is that the $B \wedge F$ term is no longer gauge invariant, as expected from the discussion above. The action that should be used in its place is

$$S_0 = \int \mathrm{Tr}(B + \mathrm{d}C) \wedge F = \int \mathrm{Tr}(B \wedge F - C \wedge \mathrm{d}F) \tag{34}$$

where the second equality holds up to a total divergence. The second term of this action looks like a magnetic monopole term in the Abelian limit. It does not have a clear interpretation for non-Abelian theories, since $\mathrm{d}F$ is not a gauge covariant object.

In any case, this choice of gauge transformation rules simplifies the vector gauge transformations quite remarkably. Now it is the combination $B + \mathrm{d}C$ which transforms homogeneously, $B' + \mathrm{d}C' = U(B + \mathrm{d}C)U^\dagger$. I can now define vector gauge transformations for the non-Abelian two-form without reference to any connection and in fact these are exactly the same as the familiar Kalb–Ramond symmetry [35],

$$B \to B + \mathrm{d}\xi \qquad C \to C - \xi. \tag{35}$$

Here ξ is a one-form which transforms homogeneously under the gauge group, i.e. the difference of two connections, $\xi \to U\xi U^\dagger$. Then $(C - \xi)$ also transforms like a connection, and the behaviour of B under $SU(N)$ gauge transformations is maintained. Therefore, just as in the previous section, the auxiliary one-form C transforms like a gauge field under ordinary $SU(N)$ gauge transformations, and shifts by the difference of two connections under Kalb–Ramond symmetry. I will show that actions invariant under the gauge transformations of (33) and the Kalb–Ramond transformations (35) are also equivalent to the actions of section 3 by a symmetry transformation.

Before discussing invariant actions, let me briefly mention one peculiarity of the (B, C) system. The combination $(B + \mathrm{d}C)$ appears to have another obvious symmetry, under $C \to C + \mathrm{d}\chi$ where χ is a scalar. This symmetry is not compatible with gauge symmetry, since $C + \mathrm{d}\chi$ cannot transform as in (33) for any choice of χ. So this symmetry is not likely to have any physical significance. However, the fields B and C must always appear in the combination $(B + \mathrm{d}C)$ and their derivatives, so this symmetry will always be present in the action.

Invariant actions involving the (B, C) pair are now easy to construct. Note that a 'field strength' $F_C = \mathrm{d}C + C \wedge C$ is covariant under $SU(N)$ gauge transformations, but is not invariant under Kalb–Ramond transformations. So it will not appear in an invariant action. Note also that even though C transforms like a gauge field, the gauge covariant derivative will be defined as d_A, since it is a good idea to leave the covariant derivative unmolested after a Kalb–Ramond transformation.

Then the actions invariant under both the gauge group and the Kalb–Ramond symmetry are already known. These are simply the actions mentioned in section 3, but with $(B + \mathrm{d}C)$ replacing $(B + \mathrm{d}A)$. The $B \wedge F$ interaction has to be replaced by the term S_0 of (34) as well. For example, the action corresponding to the parity violating action S_1 of equation (16) is

$$S_1 = \int \mathrm{Tr}\left((B + \mathrm{d}C) \wedge F - \frac{1}{2}(B + \mathrm{d}C) \wedge (B + \mathrm{d}C)\right) = \int \frac{1}{2}\,\mathrm{Tr}\, F \wedge F \tag{36}$$

where the second equality comes from substituting the equation of motion $B = F - \mathrm{d}C$ into the action. Another example is the action for the first-order formulation of Yang–Mills theory, which is now

$$S_2 = \int \mathrm{Tr}\left((B + \mathrm{d}C) \wedge F + \frac{1}{2}(B + \mathrm{d}C) \wedge *(B + \mathrm{d}C)\right) = \int \frac{1}{2}\,\mathrm{Tr}\, F \wedge *F. \tag{37}$$

Again I have substituted the equation of motion for B, which is $B = *F - \mathrm{d}C$, into the action to produce the second equality. Unlike in section 3 where actions were constructed only from the gauge field A and the two-form B, and only invariance under $SU(N)$ gauge transformations was imposed, the requirement of Kalb–Ramond symmetry rules out various combinations. For example the combination $(B - C \wedge C)$, while covariant under $SU(N)$ gauge transformations, is not invariant under the Kalb–Ramond transformation. On the other hand, just as in section 3, any constant multiple of the gauge field strength F can be added to $(B + \mathrm{d}C)$ for a gauge covariant, Kalb–Ramond invariant combination. It is easy to see that actions built with these combinations are equivalent to those already mentioned.

As mentioned earlier the gauge covariant derivative, which is needed to construct dynamical actions, should be taken to be d_A, even though C is also a gauge field, because C is not invariant under Kalb–Ramond transformations. The field strength for the two-form is constructed with the gauge covariant derivative d_A,

$$H = \mathrm{d}_A(B + \mathrm{d}C) = \mathrm{d}_A B + A \wedge \mathrm{d}C - \mathrm{d}C \wedge A \tag{38}$$

and the action for the dynamical two-form is then

$$S_3 = \int \mathrm{Tr}\left(\frac{1}{2}H \wedge *H + \frac{1}{2}F \wedge *F + B \wedge F - C \wedge \mathrm{d}F\right). \tag{39}$$

Needless to say that this action reduces to the usual Abelian action of topologically massive fields (either A or B) in the Abelian limit of vanishing structure constants. Another interesting point is that in this limit the action can be thought of as a (partial) first-order formulation, with C being a dual gauge field.

In the non-Abelian theory, C is shifted by the difference of two connections under a Kalb–Ramond transformation. I can choose the two connections to be C and A as in section 4, $\xi = \alpha(C - A)$. Then

$$B' = B + \alpha\mathrm{d}(C - A) \qquad C' = C - \alpha(C - A). \tag{40}$$

Since this is a special case of the Kalb–Ramond transformations, the combination $(B + \mathrm{d}C)$ remains invariant, as does the field strength H. For $\alpha = 1$, this has the effect of replacing C by A. Thus the actions mentioned earlier in section 3 are equivalent to the actions mentioned in this section as well.

The field strength H of equation (38) is invariant, as before, under a semiglobal transformation constructed with the help of an arbitrary local $SU(N)$ transformation matrix $\tilde{U}$. These now take a slightly different form,

$$B' = B + \alpha(A \wedge A - \tilde{\phi} \wedge \tilde{\phi}) \qquad C' = C - \alpha(A - \tilde{\phi}) \tag{41}$$

where $\tilde{\phi} = -\mathrm{d}\tilde{U}\tilde{U}^\dagger$. As before, the combination $(B + \mathrm{d}C)$ is not invariant under this transformation, but the field strength H is. Also as before, these transformations are not connected to the identity for $\tilde{\phi} \to 0$.

The conserved current for this transformation depends on the form of the interaction term. For the action of (39) the current depends on the choice of $\tilde{U}$ and is proportional to

$$\mathrm{Tr}(A \wedge A \wedge A + \tilde{\phi} \wedge F). \tag{42}$$

6. Summary of results

In this paper, I have argued that an antisymmetric tensor potential valued in the adjoint representation could have an inhomogeneous component in its gauge transformation rule, as shown in (6). It is not clear if there is a geometrical interpretation of this rule, i.e. a geometrical object which corresponds to this. But at any rate it opens up new avenues of investigation. These new gauge transformation rules are not in contradiction with theorems based on BRST analysis. In particular, the new rules do not obviate the need for an auxiliary one-form in actions containing terms quadratic in B.

I have also shown that the auxiliary one-form C transforms as a gauge field. Therefore it cannot be shifted away to zero by a vector gauge transformation, because an ordinary gauge transformation changes a vanishing gauge field to a non-vanishing flat connection. But it can be 'replaced' by the usual gauge field A via a vector gauge transformation. The actions in section 4 can be related in this way to the actions in section 3, which may be thought of as being somewhat analogous to a *unitarity gauge* choice for these theories. An outcome of this is that the action for the topological mass generation mechanism becomes gauge equivalent to a nonlinear σ-model for the gauge field, as in (21). That action also has a new kind of *semiglobal* symmetry which depends on arbitrary flat connections, but is connected to the identity only in the limit of a vanishing global parameter. This symmetry is not generated by local constraints, unlike all other known local symmetries in classical field theory. The conserved current or this symmetry is parametrized by the space of flat connections. It is in fact possible to construct a family of BRST operators, parametrized by flat connections, which anticommute with one another.

I have also shown that it is possible to define inhomogeneous gauge transformation rules for the pair of fields (B, C) without referring to the gauge field A. The actions invariant under these rules are symmetric under the usual (Abelian) Kalb–Ramond symmetry. Unlike for the vector gauge symmetry, the integral of B on a closed 2-surface is invariant under Kalb–Ramond symmetry. The actions of section 5 are also equivalent, by gauge transformations and field redefinitions, to the actions discussed in the earlier sections. The symmetries and actions mentioned in this paper should be useful for all $B \wedge F$ type theories, including gravity.

After this paper was circulated as an e-print, a set of transformations for B, similar in some respects to those found here, was found from a different approach [45], suggesting that the geometrical description for these fields may be as connections on non-Abelian gerbes [46].

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References

[1] Freedman D Z 1977 Gauge theories of antisymmetric tensor fields *Cal Tech Preprint* CALT-68-624
[2] Seo K, Okawa M and Sugamoto A 1979 *Phys. Rev.* D **19** 3744
[3] Freedman D Z and Townsend P K 1981 *Nucl. Phys.* B **177** 282
[4] Schwarz A S 1978 *Lett. Math. Phys.* **2** 247
[5] Schwarz A S 1979 *Commun. Math. Phys.* **67** 1
[6] Witten E 1988 *Commun. Math. Phys.* **117** 353
[7] Birmingham D *et al* 1991 *Phys. Rep.* **209** 129
[8] Schonfeld J F 1981 *Nucl. Phys.* B **185** 157
[9] Deser S, Jackiw R and Templeton S 1982 *Phys. Rev. Lett.* **48** 975
[10] Deser S, Jackiw R and Templeton S 1982 *Ann. Phys., NY* **140** 372
[11] Horowitz G T 1989 *Commun. Math. Phys.* **125** 417
[12] Blau M and Thompson G 1991 *Ann. Phys., NY* **205** 130
[13] Peldan P 1994 *Class. Quantum. Grav.* **11** 1087
[14] Baez J C 1994 Knots and quantum gravity: progress and prospects *Preprint* gr-qc/9410018
[15] Baez J C 2000 *Lect. Notes Phys.* **543** 25
[16] Fucito F, Martellini M and Zeni M 1997 *Nucl. Phys.* B **496** 259

[17] Cattaneo A S *et al* 1998 *Commun. Math. Phys.* **197** 571
[18] Fucito F *et al* 1997 *Phys. Lett.* B **404** 94
[19] Accardi A and Belli A 1997 *Mod. Phys. Lett.* A **12** 2353
[20] Chan H, Faridani J and Tsou S T 1995 *Phys. Rev.* D **52** 6134
[21] Chan H, Faridani J and Tsou S 1996 *Phys. Rev.* D **53** 7293
[22] Lahiri A 1992 *Phys. Lett.* B **297** 248
[23] Lahiri A 1993 Generating vector boson masses *Preprint* hep-th/9301060
[24] Cremmer E and Scherk J 1974 *Nucl. Phys.* B **72** 117
[25] Aurilia A and Takahashi Y 1981 *Prog. Theor. Phys.* **66** 693
[26] Govindarajan T R 1982 *J. Phys. G: Nucl. Phys.* **8** L17
[27] Allen T J, Bowick M J and Lahiri A 1990 *Phys. Lett.* B **237** 47
[28] Allen T J, Bowick M J and Lahiri A 1991 *Mod. Phys. Lett.* A **6** 559
[29] Minahan J A and Warner R C 1989 Stuckelberg revisited *Florida University Preprint* UFIFT-HEP-89-15
[30] Hwang D S and Lee C 1997 *J. Math. Phys.* **38** 30
[31] Lahiri A 1997 *Phys. Rev.* D **55** 5045
[32] Lahiri A 2001 *Phys. Rev.* D **63** 105002
[33] Teitelboim C 1986 *Phys. Lett.* B **167** 63
[34] Lahiri A 2001 Gauge transformations of the non-Abelian two-form *Preprint* hep-th/0107104
[35] Kalb M and Ramond P 1974 *Phys. Rev.* D **9** 2273
[36] Thierry-Mieg J and Baulieu L 1983 *Nucl. Phys.* B **228** 259
[37] Oda I and Yahikozawa S 1990 *Phys. Lett.* B **234** 69
[38] Smailagic A and Spallucci E 2000 *Phys. Lett.* B **489** 435
[39] Henneaux M 1992 On the use of auxiliary fields in classical mechanics and in field theory *Proc. 1991 Joint Summer Research Conference on Mathematical Aspects of Classical Field Theory* (ULB-TH2-91-04 in Contemporary Mathematics)
[40] Henneaux M *et al* 1997 *Phys. Lett.* B **410** 195
[41] Lee C Y and Lee D W 1998 *J. Phys. A: Math. Gen.* **31** 7809
[42] Harikumar E, Lahiri A and Sivakumar M 2001 *Phys. Rev.* D **63** 105020
[43] Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Princeton, NJ: Princeton University Press)
[44] Barnich G, Brandt F and Henneaux M 2000 *Phys. Rep.* **338** 439
[45] Chepelev I 2002 *J. High Energy Phys.* JHEP02(2002)013
[46] Breen L and Messing W 2001 Differential geometry of gerbes *Preprint* math.ag/0106083